Birational geometry of symplectic resolutions of nilpotent orbits

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1 Introduction

Let G be a complex simple Lie group and let $\mathfrak{g}$ be its Lie algebra. Then G has the adjoint action on $\mathfrak{g}$. The orbit $\mathcal{O}_x$ of a nilpotent element $x \in \mathfrak{g}$ is called a nilpotent orbit. A nilpotent orbit $\mathcal{O}_x$ admits a non-degenerate closed 2-form ω called the Kostant-Kirillov symplectic form. The closure $\bar{\mathcal{O}}_x$ of $\mathcal{O}_x$ then becomes a symplectic singularity. In other words, the 2-form ω extends to a holomorphic 2-form on a resolution of $\bar{\mathcal{O}}_x$. A resolution of $\bar{\mathcal{O}}_x$ is called a symplectic resolution if this extended form is everywhere non-degenerate on the resolution. A typical symplectic resolution of $\bar{\mathcal{O}}_x$ is obtained as the *Springer resolution*

$$T^*(G/P) \to \bar{\mathcal{O}}_x$$

for a suitable parabolic subgroup $P \subset G$. Here $T^*(G/P)$ is the cotangent bundle of the homogenous space G/P. Spaltenstein [S] and Hesselink [He] obtained a necessary and sufficient condition for $\bar{\mathcal{O}}_x$ to have a Springer resolution when $\mathfrak{g}$ is a classical simple Lie algebra. Moreover, [He] gave an explicit number of such parabolics P up to conjugacy class that give Springer resolutions of $\bar{\mathcal{O}}_x$ (cf. §2). Recently, Fu [Fu 1] has shown that every symplectic (projective) resolution is obtained as a Springer resolution. The following is one of main results of this paper.

Theorem 4.4. *Let $\mathcal{O}_x$ be a nilpotent orbit of a classical complex simple Lie algebra. Let Y and Y' be any two Springer resolutions of the closure $\bar{\mathcal{O}}_x$ of the nilpotent orbit. Then the birational map $Y - - \to Y'$ can be decomposed into finite number of diagrams $Y_i \to X_i \leftarrow Y_{i+1}$ $(i = 1, ..., m-1)$ with $Y_1 = Y$ and $Y_m = Y'$ in such a way that each diagram is locally a trivial family of Mukai flops of type A or of type D.*

A *Mukai flop of type A* is a kind of Springer resolutions; let $x \in \mathfrak{sl}(n)$ be a nilpotent element of type $[2^k, 1^{n-2k}]$ with $2k < n$. Then a Mukai flop of type A is the diagram of two Springer resolutions of $\bar{\mathcal{O}}_x$:

$$T^*G(k,n) \to \bar{\mathcal{O}}_x \leftarrow T^*G(n-k,n)$$

where $G(k,n)$ (resp. $G(n-k,n)$) is the Grassmannian which parametrizes k-dimensional (resp. $n-k$-dimensional) subspaces of $\mathbf{C}^n$. This flop naturally appears in the wall-crossing of the moduli spaces of various objects (eg. stable sheaves on K3 surfaces, quiver varieties and so on). On the other hand, a *Mukai flop of type D* comes from an orbit of a simple Lie algebra of type D. Let $x \in \mathfrak{so}(2k)$ be a nilpotent element of type $[2^{k-1}, 1^2]$, where k is an odd integer with $k \geq 3$. Then $\bar{\mathcal{O}}_x$ admits two Springer resolutions

$$T^*G^+_{iso}(k,2k) \to \bar{\mathcal{O}}_x \leftarrow T^*G^-_{iso}(k,2k)$$

where $G^+_{iso}(k,2k)$ and $G^-_{iso}(k,2k)$ are two connected components of the orthogonal Grassmannian $G_{iso}(k,2k)$. For details on these Mukai flops, see §3. The factorizations in Theorem 4.4 make it possible to describe the ample cones and movable cones of symplectic resolutions of $\bar{\mathcal{O}}_x$. Moreover, Theorem 4.4 would clarify the geometric meaning of the results of Spaltenstein and Hesselink. To illustrate these, three examples will be given (see Examples 4.6, 4.7, 4.8).

Another purpose of this paper is to give an affimative answer to the following conjecture in the case of (the normalization of) a nilpotent orbit closure in a classical simple Lie algebra (Theorem 5.9).

Conjecture([F-N]): *Let W be a normal symplectic singularity. Then for any two symplectic resolutions $f_i : X_i \to W$, $i = 1,2$, there are deformations $\mathcal{X}_i \xrightarrow{F_i} \mathcal{W}$ of f_i over a parameter space S such that, for $s \in S - \{0\}$, $F_{i,s} : \mathcal{X}_{i,s} \to \mathcal{W}_s$ are isomorphisms. In particular, X_1 and X_2 are deformation equivalent.*

This conjecture is already proved in [F-N] when W is a nilpotent orbit closure in $\mathfrak{sl}(n)$. On the other hand, a weaker version of this conjecture is proved in [Fu 2] when W is the normalization of a nilpotent orbit closure in a classical simple Lie algebra. According to the idea of Borho and Kraft [B-K], we shall define a deformation of $\bar{\mathcal{O}}_x$ by using a *Dixmier sheet.* Corresonding to each parabolic subgroup P, this deformation has a simultaneous resolution. These simultaneous resolutions would give the desired deformations of the conjecture. Details on the construction of them can be found in §5.

Notation and Convention. A partition $\mathbf{d}$ of n is a set of positive integers $[d_1, ..., d_k]$ such that $\Sigma d_i = n$ and $d_1 \geq d_2 \geq ... \geq d_k$. We mean by $[d_1^{j_1}, ..., d_k^{j_k}]$ the partition where d_i appear in j_i multiplicity. If $(p_1, ..., p_s)$ is a sequence of positive integers, then we define the partition $\mathbf{d} = \mathrm{ord}(p_1, ..., p_s)$ by $d_i := \sharp\{j; p_j \geq i\}$. In particular, for a partition $\mathbf{d}$, ${}^t\mathbf{d} := \mathrm{ord}(d_1, ..., d_k)$ is called the dual partition of $\mathbf{d}$. We define $d^i := ({}^t d)_i$.

2 Nilpotent orbits and Polarizations

Let G be a complex simple Lie group and let $\mathfrak{g}$ be its Lie algebra. G has the adjoint action on $\mathfrak{g}$. The orbit $\mathcal{O}_x$ of a nilpotent element $x \in \mathfrak{g}$ for this action is called a nilpotent orbit. This orbit carries a natural closed non-degenerate 2-form (Kostant-Kirillov form) ω (cf. [C-G], Prop. 1.1.5, [C-M], 1.3), and its closure $\bar{\mathcal{O}}_x$ becomes a *symplectic* singularity, that is, the symplectic 2-form ω extends to a holomorphic 2-form on a resolution Y of $\bar{\mathcal{O}}_x$. When $\mathfrak{g}$ is classical, $\mathfrak{g}$ is naturally a Lie subalgebra of $\mathrm{End}(V)$ for a complex vector space V. Then we can attach a partition $\mathbf{d}$ of $n := \dim V$ to each orbit as the Jordan type of an element contained in the orbit. Here a partition $\mathbf{d} := [d_1, d_2, ..., d_k]$ of n is a set of positive integers with $\Sigma d_i = n$ and $d_1 \geq d_2 \geq ... \geq d_k$. When a number e appears in the partition $\mathbf{d}$, we say that e is a *part* of $\mathbf{d}$. We call $\mathbf{d}$ *very even* when $\mathbf{d}$ consists with only even parts, each having even multiplicity. The following result can be found, for example, in [C-M, §5].

Proposition 2.1. *Let $\mathcal{N}(\mathfrak{g})$ be the set of nilpotent orbits of $\mathfrak{g}$.*

(1)(A_{n-1}): When $\mathfrak{g} = \mathfrak{sl}(n)$, there is a bijection between $\mathcal{N}(\mathfrak{g})$ and the set of partitions $\mathbf{d}$ of n.

(2)(B_n): When $\mathfrak{g} = \mathfrak{so}(2n+1)$, there is a bijection between $\mathcal{N}(\mathfrak{g})$ and the set of partitions $\mathbf{d}$ of $2n+1$ such that even parts occur with even multiplicity.

(3)(C_n): When $\mathfrak{g} = \mathfrak{sp}(2n)$, there is a bijection between $\mathcal{N}(\mathfrak{g})$ and the set of partitions $\mathbf{d}$ of $2n$ such that odd parts occur with even multiplicity

(4)(D_n): When $\mathfrak{g} = \mathfrak{so}(2n)$, there is a surjection f from $\mathcal{N}(\mathfrak{g})$ to the set of partitions $\mathbf{d}$ of $2n$ such that even parts occur with even multiplicity. For a partition $\mathbf{d}$ which is not very even, $f^{-1}(\mathbf{d})$ consists of exactly one orbit, but, for very even $\mathbf{d}$, $f^{-1}(\mathbf{d})$ consists of exactly two different orbits.

Let V be a finite dimensional $\mathbf{C}$-vector space. Then a flag $F := \{F_i\}_{1 \leq i \leq s}$

is a sequence of vector subspaces of V:

$$F_1 \subset F_2 \subset ... \subset F_s = V.$$

We put $p_i := \dim(F_i/F_{i-1})$ and the sequence $(p_1, ..., p_s)$ is called the type of F. The following is well-known (cf. [He], Lemma 4.3):

Proposition 2.2. *Every parabolic subgroup P of $SL(V)$ is a stabilizer of a suitable flag F of V, and conversely, for every flag F of V, its stabilizer group $P \subset SL(V)$ is a parabolic subgroup. The conjugacy class of a given parabolic subgroup is completely determined by the type of the corresponding flag.*

Let ϵ denote the number 0 or 1. Assume that V is a **C**-vector space equipped with a non-degenerate bilinear form $<, >$ such that

$$< v, w >= (-1)^{\epsilon} < w, v >, (v, w \in V).$$

When $\epsilon = 0$ (resp. $\epsilon = 1$), this means that the bilinear form is symmetric (resp. skew-symmetric). We put

$$H := \{x \in GL(V); < xv, xw >=< v, w >, (v, w \in V)\},$$

and

$$G := \{x \in H; \det(x) = 1\}.$$

Note that

$$H = \begin{cases} O(V) & (\epsilon = 0) \\ Sp(V) & (\epsilon = 1) \end{cases}$$

and

$$G = \begin{cases} SO(V) & (\epsilon = 0) \\ Sp(V) & (\epsilon = 1) \end{cases}$$

A flag $F := \{F_i\}_{1 \le i \le s}$ of V is called *isotropic* if $F_i^{\perp} = F_{s-i}$ for $1 \le i \le s$. An isotropic flag F is *admissible* if the stabilizer group P of F has no finner stabilized flag than F. In other words, let $P_F := \{g \in G; gF_i \subset F_i \forall i\}$. Then, for any i, there is no P_F-invariant subspace F'_i such that $F_i \subset F'_i \subset F_{i+1}$ with $F'_i \neq F_i, F_{i+1}$. When the length s of an isotropic flag F is even, one can write the type of F as $(p_1, ..., p_k, p_k, ..., p_1)$ with $k = s/2$. On the other hand, when s is odd, one can write the type of F as $(p_1, ..., p_k, q, p_k, ..., p_1)$ with $k = (s-1)/2$.

Example 2.3. *Assume that $\epsilon = 0$ and $q = 2$. Then one can always find a P_F-invariant subspace F'_k such that $F_k \subset F'_k \subset F_{k+1}$ and $\dim(F'_k/F_k) = 1$. Therefore, F is an isotropic flag which is not admissible. This is the only case where an isotropic flag is not admissible.*

Proposition 2.4. *Let V be a finite dimensional $\mathbf{C}$-vector space with a non-degenerate bilinear form and let ϵ be 0 or 1 according as the bilinear form is symmetric or skew-symmetric. Let G and H be the same as above.*

(i) Any parabolic subgroup P of G is a stabilizer group of an admissible flag of V, and conversely, the stabilizer group of an admissible flag becomes a parabolic subgroup of G.

(ii) There is a one-to-one correspondence between H-conjugacy classes of parabolic subgroups of G, and types of the stabilized admissible flags.

(iii) When $\epsilon = 1$, the H-conjugacy class of a parabolic subgroup of G coincides with the G-conjugacy class of a parabolic subgroup of G.

(iv) When $\epsilon = 0$, the H-conjugacy class of a parabolic subgroup of G coincides with the G-conjugacy class of a parabolic subgroup of G except in the case where the type of the stabilized flag satisfies $q = 0$ and $p_k \geq 2$ (cf. the notation above). In this particular case, the H-conjugacy class splits into two G-conjugacy classes.

Definition 1. *If a parabolic subgroup of G has a stabilized (admissible) flag F of type $(p_1, ..., p_k, q, p_k, ..., p_1)$, then $\pi := \mathrm{ord}(p_1, ..., p_k, q, p_k, ..., p_1)$ is called the* Levi type *of P.*

Remark 2.5. *Assume that $\epsilon = 0$ and $\dim V$ is even. Put $p = 1/2 \dim V$ and let $G_{\mathrm{iso}}(p, V)$ be the* orthogonal Grassmannian *which parametrizes isotropic p dimensional subspaces of V. The orthogonal Grassmannian has two connected components. Take two points $[F_1]$, $[F_2]$ from different components of $G_{\mathrm{iso}}(p, V)$. Then one can find $h \in H$ such that $h[F_1] = [F_2]$, but, when $p \geq 2$, there are no such elements in G. This is the reason why the exceptional cases occur in (iv) of the proposition*

Let G be a complex simple Lie algebra as above and let $x \in \mathfrak{g}$ be a nilpotent element. Then a parabolic subgroup P of G is called a *polarization* of x if $x \in \mathfrak{n}(P)$ and $\dim \mathcal{O}_x = 2 \dim G/P$, where $\mathfrak{n}(P)$ is the nil-radical of $\mathfrak{p} := \mathrm{Lie}(P)$. If we have a polarization P of x, then we can define a map

$$\mu : G \times_P \mathfrak{n}(P) \to \bar{\mathcal{O}}_x$$

as $\mu([g, x]) = \mathrm{Ad}_g(x)$. Here $G \times_P \mathfrak{n}(P)$ is the quotient space of $G \times \mathfrak{n}(P)$ by the equivalence relation:

$$(g, x) \sim (g', x') \Leftrightarrow g' = gp, x' = \mathrm{Ad}_{p^{-1}}(x), \exists p \in P.$$

$G \times_P \mathfrak{n}(P)$ is a vector bundle over G/P and it coincides with the cotangent bundle $T^*(G/P)$ of G/P. The map μ is a generically finite, proper and surjective map. If $\deg(\mu) = 1$, then μ becomes a resolution of $\bar{\mathcal{O}}_x$. In this case, μ is called the *Springer resolution* of $\bar{\mathcal{O}}_x$ with respect to P.

Let $x \in \mathfrak{g}$ be a nilpotent orbit and denote by $\mathrm{Pol}(x)$ the set of polarizations of x.

Theorem 2.6. *Let $x \in \mathfrak{sl}(n)$ be a nilpotent element. Then* $\mathrm{Pol}(x) \neq \emptyset$*. Assume that x is of type $\mathbf{d} = [d_1, ..., d_k]$. Then $P \in \mathrm{Pol}(x)$ has the flag type $(p_1, ..., p_s)$ such that* $\mathrm{ord}(p_1, ..., p_s) = \mathbf{d}$*. Conversely, for any sequence $(p_1, ..., p_s)$ with* $\mathrm{ord}(p_1, ..., p_s) = \mathbf{d}$*, there is a unique polarization $P \in \mathrm{Pol}(x)$ which has the flag type $(p_1, ..., p_s)$.*

Proof. We shall construct a flag F of type $(p_1, ..., p_s)$ such that $xF_i \subset F_{i-1}$ for all i. We identify the partition $\mathbf{d}$ with a Young table consisting of n boxes, where the i-th row consists of d_i boxes for each i. We denote by (i, j) the box of $\mathbf{d}$ lying on the i-th row and on the j-th column. Let $e(i, j)$, $(i, j) \in \mathbf{d}$ be a *Jordan basis* of $V := \mathbf{C}^n$ such that $xe(i, j) = e(i-1, j)$. We consruct a flag by the induction on n. Define first $F_1 := \Sigma_{1 \leq j \leq p_1} \mathbf{C}e(1, j)$. Then x induces a nilpotent endomorphism $\bar{x}$ of V/F_1. The Jordan type of $\bar{x}$ is $[d_1 - 1, ..., d_{p_1} - 1, d_{p_1+1}, ..., d_k]$. Note that this coincides with $\mathrm{ord}(p_2, ..., p_k)$. By the induction hypothesis, we already have a flag of type $(p_2, ..., p_k)$ on V/F_1 stabilized by $\bar{x}$; hence we have a desired flag F. Let P be the stabilizer group of F. Then it is clear that $x \in \mathfrak{n}(P)$. By an explicit calculation $\dim \mathcal{O}_x = 2 \dim G/P$.

Next consider simple Lie algebras of type B, C or D. Let V be an n dimensional $\mathbf{C}$-vector space with a non-degenerate symmetric (skew-symmetric) form. As above, $\epsilon = 0$ when this form is symmetric and $\epsilon = 1$ when this form is skew-symmetric. Let $P_\epsilon(n)$ be the set of partitions $\mathbf{d}$ of n such that $\sharp\{i; d_i = m\}$ is even for every integer m with $m \equiv \epsilon \pmod 2$. Note that these partitions are nothing but those which appear as the Jordan types of nilpotent elements of $\mathfrak{so}(n)$ or of $\mathfrak{sp}(n)$. Next, let q be a non-negative integer and assume moreover that $q \neq 2$ when $\epsilon = 0$. We define $\mathrm{Pai}(n, q)$ to be the set of partitions π of n such that $\pi_i \equiv 1$ (mod 2) if $i \leq q$ and $\pi_i \equiv 0$ (mod

2) if $i > q$. Note that, if $(p_1, ..., p_k, q, p_k, ..., p_1)$ is the type of an admissible flag of V, then $\mathrm{ord}(p_1, ..., p_k, q, p_k, ..., p_1) \in \mathrm{Pai}(n, q)$. Now we shall define the *Spaltenstein map* S from $\mathrm{Pai}(n, q)$ to $P_\epsilon(n)$. For $\pi \in \mathrm{Pai}(n, q)$, let

$$I(\pi) := \{j \in \mathbf{N} | j \not\equiv n \,(\mathrm{mod}\, 2), \pi_j \equiv \epsilon \,(\mathrm{mod}\, 2), \pi_j \geq \pi_{j+1} + 2\}.$$

Then the Spaltenstein map

$$S : \mathrm{Pai}(n, q) \to P_\epsilon(n)$$

is defined as

$$S(\pi)_j := \begin{cases} \pi_j - 1 & (j \in I(\pi)) \\ \pi_j + 1 & (j - 1 \in I(\pi)) \\ \pi_j & (\mathit{otherwise}) \end{cases}$$

Theorem 2.7. *Let G be $SO(V)$ or $Sp(V)$ according as $\epsilon = 0$ or $\epsilon = 1$. Let $x \in \mathfrak{g}$ be a nilpotent element of type $\mathbf{d} \in P_\epsilon(n)$. For $\pi \in \mathrm{Pai}(n, q)$, define $\mathrm{Pol}(x, \pi)$ to be the set of polarizations of x with Levi type π (cf. Definition 1). Then $\mathrm{Pol}(x, \pi) \neq \emptyset$ if and only if $S(\pi) = \mathbf{d}$.*

Proof. The proof of this theorem can be found in [He], Theorem 7.1, (a). But we prove here that $\mathrm{Pol}(x, \pi) \neq \emptyset$ if $S(\pi) = \mathbf{d}$ because we will later use this argument. There is a basis $\{e(i, j)\}$ of V indexed by the Young diagram $\mathbf{d}$ with the following properties (cf. [S-S], p.259, see also [C-M], 5.1.)

(i) $\{e(i, j)\}$ is a Jordan basis of x, that is, $xe(i, j) = e(i-1, j)$ for $(i, j) \in \mathbf{d}$.

(ii) $< e(i, j), e(p, q) > \neq 0$ if and only if $p = d_j - i + 1$ and $q = \beta(j)$, where β is a permutation of $\{1, 2, ..., d^1\}$ which satiesfies: $\beta^2 = id$, $d_{\beta(j)} = d_j$, and $\beta(j) \not\equiv j \pmod 2$ if $d_j \not\equiv \epsilon \pmod 2$. One can choose an arbitrary β within these restrictions.

For a sequence $(p_1, ..., p_s)$ with $\pi = \mathrm{ord}(p_1, ..., p_s)$ and $p_i = p_{s+1-i}, (1 \leq i \leq s)$, we shall construct an admissible flag F of type $(p_1, ..., p_s)$ such that $xF_i \subset F_{i-1}$ for all i. We proceed by the induction on s. When $s = 1$, $\pi = [1^n]$ and $\pi = \mathbf{d}$. In this case, $x = 0$ and F is a trivial flag $F_1 = V$. When $s > 1$, we shall construct an isotropic flag $0 \subset F_1 \subset F_{s-1} \subset V$. Put $p := p_1 (= p_s)$ and let $\rho := \mathrm{ord}(p_2, ..., p_{s-1}) \in \mathrm{Pai}(n - 2p, q)$. Then we have

$$\rho_j := \begin{cases} \pi_j - 2 & (j \leq p) \\ \pi_j & (j > p) \end{cases}$$

Let

$$S' : \mathrm{Pai}(n-2p, q) \to P_\epsilon(n-2p)$$

be the Spatenstein map and we put $\mu := S'(\rho)$. There are two cases (A) and (B). The first case (A) is when $i(\pi) = \{p\} \cup I(\rho)$ and $p \notin I(\rho)$. In this case, $p \not\equiv n \pmod 2$, $\pi_p \equiv \epsilon \pmod 2$ and $\pi_p = \pi_{p+1} + 2$. Now we have

$$\mu_j = d_j - 2, (j < p),$$

$$\mu_p = d_p - 1,$$

$$\mu_{p+1} = d_{p+1} - 1,$$

$$\mu_j = d_j, (j > p+1),$$

where $d_p = d_{p+1}$. The second case is exactly when (A) does not occur. In this case, $I(\pi) = I(\rho)$ and

$$\mu_j = d_j - 2, (j \leq p),$$

$$\mu_j = d_j, (j > p).$$

Let us assume that the case (A) occurs. We choose the basis $e(i,j)$ of V in such a way that the permutaion β satisfies $\beta(p) = p+1$. There are two choices for F_1. The first one is to put

$$F_1 = \Sigma_{1 \leq j \leq p} \mathbf{C} e(1,j).$$

The second one is to put

$$F_1 = \Sigma_{1 \leq j \leq p+1, j \neq p} \mathbf{C} e(1,j).$$

In any case, we put $F_{s-1} = F_1^{\perp}$. Then x induces a nilpotent endomorphism of F_{s-1}/F_1 of type μ. Next assume that the case (B) occurs. In this case, we put

$$F_1 = \Sigma_{1 \leq j \leq p} \mathbf{C} e(1,j)$$

and $F_{s-1} = F_1^{\perp}$. Then x induces a nilpotent endomorphism of F_{s-1}/F_1 of type μ. By the induction on s, we have an admissible filtration $0 \subset F_1 \subset ... \subset F_{s-1} \subset V$ with desired properties. Let P be the stabilizer group of F. Then it is clear that $x \in \mathfrak{n}(P)$. By an explicit calculation $\dim \mathcal{O}_x = 2 \dim G/P$.

Theorem 2.8. *Let G and $\mathfrak{g}$ be the same as Theorem 2.7. Let $x \in \mathfrak{g}$ be a nilpotent element of type* $\mathbf{d}$ *and denote by $\mathcal{O}$ the orbit containing x. Assume that P is a polarization of x with Levi type π. Let*

$$\mu : T^*(G/P) \to \bar{\mathcal{O}}$$

be the Springer map. Then

$$\deg(\mu) := \begin{cases} 2^{\sharp I(\pi)-1} & (q = \epsilon = 0, \pi^i \not\equiv 0 \,(\mathrm{mod}\, 2)\, \exists i) \\ 2^{\sharp I(\pi)} & (q + \epsilon \geq 1 \text{ or } q = \epsilon = 0,\ \pi^i \equiv 0 \,(\mathrm{mod}\, 2)\, \forall i) \end{cases}$$

Moreover, if $\deg(\mu) = 1$*, then the Levi type of P is unique. In other words, if two polarizations of x respectively give Springer resolutions of $\bar{\mathcal{O}}$, then they have the same Levi type.*

Proof. The first part is [He], Theorem 7.1, (d) (cf. [He], §1). The proof of the second part is rather technical, but for the completeness, we include it here. Let

$$B(\mathbf{d}) = \{j \in \mathbf{N}; d_j > d_{j+1}, d_j \not\equiv \epsilon \,(\mathrm{mod}\, 2)\}.$$

Note that $S(\pi) = \mathbf{d}$, where S is the Spaltenstein map. When $\epsilon = 0$, $B(\mathbf{d}) = \emptyset$ if and only if $q = 0$ and $d^i \equiv 0$ (mod 2) for all i. Assume that $B(\mathbf{d}) = \emptyset$. Since $\deg(\mu) = 1$, by the first part of our theorem, $\sharp I(\pi) = 0$. Then $\pi = \mathbf{d}$. Assume that $B(\mathbf{d}) \neq \emptyset$. If $q \neq 0$ for our π or $\epsilon = 1$, then $\sharp I(\pi) = 0$; hence $\pi = \mathbf{d}$. If $\epsilon = 0$ and $q = 0$ for π, then $\sharp I(\pi) = 1$. Since $\sharp I(\pi) = 1/2\, \sharp\{j; d_j \equiv 1 \,(\mathrm{mod}\, 2)\}$ by [He], Lemma 6.3, (b). This implies that $\sharp\{j; d_j \equiv 1 \,(\mathrm{mod}\, 2)\} = 2$. Note that π with $q = 0$ is uniquely determined by $\mathbf{d}$ because the Spaltenstein map is injective ([He], Prop. 6.5, (a)).

Now let us prove the second part of our theorem. When $\epsilon = 1$, we should have $\pi = \mathbf{d}$ by the argument above. Next consider the case where $\epsilon = 0$. Assume that there exist two polarizations P_1 and P_2 giving Springer resolutions. Let π_1 and π_2 be their Levi types. Assume that $\pi_1 \in \mathrm{Pai}(n, 0)$ and $\pi_2 \in \mathrm{Pai}(n, q_2)$ with $q_2 > 0$. By the argument above, we see that $\sharp\{j; d_j \equiv 1 \,(\mathrm{mod}\, 2)\} = 2$. On the other hand, since $q_2 > 0$, $\pi_2 = \mathbf{d}$. This shows that $q_2 = 2$; but, when $\epsilon = 0$, $q_2 \neq 2$ by Example 2.3, which is a contradiction. Hence, in this case, π is also uniquely determined by $\mathbf{d}$.

3 Mukai flops

Mukai flop of type A. Let $x \in \mathfrak{sl}(n)$ be a nilpotent element of type $[2^k, 1^{n-2k}]$ and let $\mathcal{O}$ be the nilpotent orbit containing x. By Theorem 2.6,

there are two polarizations P and P' of x, where P has the flag type $(k, n-k)$ and P' has the flag type $(n-k, k)$. The closure $\bar{\mathcal{O}}$ of $\mathcal{O}$ admits two Springer resolutions

$$T^*(SL(n)/P) \xrightarrow{\pi} \bar{\mathcal{O}} \xleftarrow{\pi'} T^*(SL(n)/P').$$

Note that $SL(n)/P$ is isomorphic to the Grassmannian $G(k, n)$ and $SL(n)/P'$ is isomorphic to $G(n-k, n)$.

Lemma 3.1. *When $k < n/2$, π and π' are both small birational maps and the diagram becomes a flop.*

Proof. The closure $\bar{\mathcal{O}}$ consists of finite number of orbits $\{\mathcal{O}_{[2^i,1^{n-2i}]}\}_{0\leq i\leq k}$. The main orbit $\mathcal{O}_{[2^k,1^{n-2k}]}$ is an open set of $\bar{\mathcal{O}}$. A fiber of π (resp. π') over a point of $\mathcal{O}_{[2^i,1^{n-2i}]}$ is isomorphic to the Grassmannian $G(k-i, n-2i)$ (resp. $G(n-i-k, n-2i)$). By a simple dimension count, if $k < n/2$, then π and π' are both small birational maps. Next let us prove that the diagram is a flop. Let $\tau \subset \mathcal{O}^{\oplus n}_{G(k,n)}$ (resp. $\tau' \subset \mathcal{O}^{\oplus n}_{G(n-k,n)}$) be the universal subbundle. Denote by T (resp. T') the pull-back of τ (resp. τ') by the projection $T^*G(k, n) \to G(k, n)$ (resp. $T^*G(n-k, n) \to G(n-k, n)$). We shall describe the strict transform of $\wedge^k T$ by the birational map $T^*G(k, n) - - \to T^*G(n-k, n)$. Take a point $y \in \mathcal{O}_{[2^k,1^{n-2k}]}$. Note that $T^*G(k, n)$ is naturally embedded in $G(k, n) \times \bar{\mathcal{O}}$. Then the fiber $\pi^{-1}(y)$ consists of one point $([\mathrm{Im}(y)], y) \in G(k, n) \times \bar{\mathcal{O}}$. The fiber $T_{\pi^{-1}(y)}$ of the vector bundle T over $\pi^{-1}(y)$ coincides with the vector space $\mathrm{Im}(y)$. Hence $(\wedge^k T)_{\pi^{-1}(y)}$ is isomorphic to $\wedge^k \mathrm{Im}(y)$. Now let L be the strict transform of $\wedge^k T$ by $T^*G(k, n) - - \to T^*G(n-k, n)$. First note that $(\pi')^{-1}(y)$ also consists of one point $([\mathrm{Ker}(y)], y) \in G(n-k, n) \times \bar{\mathcal{O}}$. Then, by definition, $L_{(\pi')^{-1}(y)} = \wedge^k \mathrm{Im}(y)$. Since $\wedge^k \mathrm{Im}(y) \cong (\wedge^{n-k}\mathrm{Ker}(y))^*$, we see that $L \cong (\wedge^{n-k}T')^{-1}$. Now $\wedge^k T$ is a negative line bundle. On the other hand, its strict transform L becomes an ample line bundle. This implies that our diagram is a flop.

Remark 3.2. *When $k = n/2$, π and π' are both divisorial birational contraction maps. Moreover, two resolutions are isomorphic.*

Definition 2. *The diagram*

$$T^*(SL(n)/P) \xrightarrow{\pi} \bar{\mathcal{O}} \xleftarrow{\pi'} T^*(SL(n)/P')$$

is called a (stratified) Mukai flop of type A when $k < n/2$.

Mukai flop of type D. Assume that k is an odd integer with $k \geq 3$. Let V be a $\mathbf{C}$-vector space of $\dim 2k$ with a non-degenerate symmetric form $<,>$. Let $x \in \mathfrak{so}(V)$ be a nilpotent element of type $[2^{k-1}, 1^2]$ and let $\mathcal{O}$ be the nilpotent orbit containing x. Let $S : \mathrm{Pai}(2k, 0) \to P_\epsilon(2k)$ be the Spaltenstein map, where $\epsilon = 0$ in our case. Then, for $\pi := (2^k) \in \mathrm{Pai}(2k, 0)$, $S(\pi) = [2^{k-1}, 1^2]$. Let us recall the construction of the stabilized flags by the polarizations of x in the proof of Theorem 2.7. Since $I(\pi) = \{k\}$, the case (A) occurs (cf. the proof of Theorem 2.7.); hence there are two choices of the flags. We denote by P^+ the stabilizer subgroup of $SO(V)$ of one flag, and denote by P^- the stabilizer subgroup of another one. Let $G_{iso}(k, V)$ be the orthogonal Grassmannian which parametrizes k dimensional isotropic subspaces of V. $G_{iso}(k, V)$ has two connected components ${G^+}_{iso}(k, V)$ and ${G^-}_{iso}(k, V)$. Note that $SO(V)/P^+ \cong {G^+}_{iso}(k, V)$ and $SO(V)/P^- \cong {G^-}_{iso}(k, V)$. The closure $\bar{\mathcal{O}}$ of $\mathcal{O}$ admits two Springer resolutions

$$T^*(SO(V)/P^+) \xrightarrow{\pi^+} \bar{\mathcal{O}} \xleftarrow{\pi^-} T^*(SO(V)/P^-).$$

Lemma 3.3. *π^+ and π^- are both small birational maps and the diagram becomes a flop.*

Proof. The closure $\bar{\mathcal{O}}$ consists of the orbits $\{\mathcal{O}_{[2^{k-2i-1}, 1^{4i+2}]}\}_{1 \leq i \leq 1/2(k-1)}$. The main orbit is an open set of $\bar{\mathcal{O}}$. A fiber of π^+ (resp. π^-) over a point of $\mathcal{O}_{[2^{k-2i-1}, 1^{4i+2}]}$ is isomorphic to ${G^+}_{iso}(2i+1, 4i+2)$ (resp. ${G^-}_{iso}(2i+1, 4i+2)$). By dimension counts of each orbit and of each fiber, we see that π^+ and π^- are both small birational maps. Next let us prove that the diagram is a flop. Let $\tau^+ \subset \mathcal{O}^{\oplus 2k}_{{G^+}_{iso}(k,V)}$ (resp. $\tau^- \subset \mathcal{O}^{\oplus 2k}_{{G^-}_{iso}(k,V)}$) be the universal subbundle. Denote by T^+ (resp. T^-) the pull-back of τ^+ (resp. τ^-) by the projection $T^*({G^+}_{iso}(k, V)) \to {G^+}_{iso}(k, V)$ (resp. $T^*({G^-}_{iso}(k, V)) \to {G^-}_{iso}(k, V)$). We shall describe the strict transform of $\wedge^k T^-$ by the birational map $T^*({G^-}_{iso}(k, V)) -- \to T^*({G^+}_{iso}(k, V))$. Take a point $y \in \mathcal{O}_{[2^{k-1}, 1^2]}$. Let $g \in SO(V)$ be an element such that $gxg^{-1} = y$. Note that $T^*({G^+}_{iso}(k, V))$ (resp. $T^*({G^-}_{iso}(k, V))$) is naturally embedded in ${G^+}_{iso}(k, V) \times \bar{\mathcal{O}}$ (resp. ${G^-}_{iso}(k, V) \times \bar{\mathcal{O}}$). Then the fiber $(\pi^+)^{-1}(y)$ (resp. $(\pi^-)^{-1}(y)$) consists of one point $([F_y^+], y) \in {G^+}_{iso}(k, V) \times \bar{\mathcal{O}}$ (resp. $([F_y^-], y) \in {G^-}_{iso}(k, V) \times \bar{\mathcal{O}}$) where $F_y^+ \subset V$ (resp. $F_y^- \subset V$) is the flag stabilized by gP^+g^{-1} (resp. gP^-g^{-1}). Note that gP^+g^{-1} and gP^-g^{-1} are both polarizations of y. Let us recall the construction of flags in the proof of Theorem 2.7. For y we choose a Jordan basis $\{e(i, j)\}$ of V as in the proof of Theorem 2.7. Since $\mathbf{d} = [2^{k-1}, 1^2]$, β is a

permutation of $\{1, 2, ..., k, k+1\}$. But it preserves the subsets $\{1, 2, ..., k-1\}$ and $\{k, k+1\}$ respectively. We assume that $\beta(k) = k+1$ and $\beta(k+1) = k$. In our situation, the case (A) occurs. There are two choices of the flags:

$$\Sigma_{1 \leq j \leq k-1} \mathbf{C} e(1, j) + \mathbf{C} e(1, k)$$

and

$$\Sigma_{1 \leq j \leq k-1} \mathbf{C} e(1, j) + \mathbf{C} e(1, k+1).$$

Note that one of these is stabilized by gP^+g^{-1} and another one is stabilized by gP^-g^{-1}. We may assume that

$$F_y^+ = \Sigma_{1 \leq j \leq k-1} \mathbf{C} e(1, j) + \mathbf{C} e(1, k),$$

and

$$F_y^- = \Sigma_{1 \leq j \leq k-1} \mathbf{C} e(1, j) + \mathbf{C} e(1, k+1).$$

Since $\mathrm{Ker}(y) = \Sigma_{1 \leq j \leq k+1} e(1, j)$ and $\mathrm{Im}(y) = \Sigma_{1 \leq j \leq k-1} e(1, j)$, we have two exact sequences

$$0 \to \mathrm{Ker}(y)/F_y^+ \to V/F_y^+ \to \mathrm{Im}(y) \to 0,$$

and

$$0 \to \mathrm{Im}(y) \to F_y^- \to F_y^-/\mathrm{Im}(y) \to 0.$$

Since $F_y^-/\mathrm{Im}(y) \cong \mathrm{Ker}(y)/F_y^+$, we conclude that

$$\wedge^k F_y^- \cong \wedge^k (V/F_y^+).$$

Let L be the strict transform of $\wedge^k T^-$ by the birational map $T^*(G^-{}_{iso}(k, V)) - - \to T^*(G^+{}_{iso}(k, V))$. The fiber $T^-_{(\pi^-)^{-1}(y)}$ of the vector bundle T^- is isomorphic to the vector space $\wedge^k F_y^-$. Hence, by the definition of L, $L_{(\pi^+)^{-1}(y)} = \wedge^k F_y^-$. By the observation above, we see that $L_{(\pi^+)^{-1}(y)} = \wedge^k (V/F_y^+)$. This shows that $L \cong (\wedge^k T^+)^{-1}$. Now $\wedge^k T^-$ is a negative line bundle. On the other hand, its strict transform L is an ample line bundle. This implies that our diagram is a flop.

Definition 3. *The diagram*

$$T^*(SO(V)/P^+) \xrightarrow{\pi^+} \bar{\mathcal{O}} \xleftarrow{\pi^-} T^*(SO(V)/P^-)$$

is called a (stratified) Mukai flop of type D or called an orthogonal Mukai flop.

Remark 3.4. *When k is an even integer with $k \geq 2$, there are two nilpotent orbits $\mathcal{O}^+$ and $\mathcal{O}^-$ with Jordan type $[2^k]$. They have Springer resolutions*

$$T^*(G^+{}_{iso}(k, 2k)) \to \bar{\mathcal{O}}^+,$$

and

$$T^*(G^-{}_{iso}(k, 2k)) \to \bar{\mathcal{O}}^-.$$

These resolutions are both divisorial birational contraction maps. When $k = 1$, three varieties $T^*(G^+{}_{iso}(1, 2))$, $T^*(G^-{}_{iso}(1, 2))$ and $\bar{\mathcal{O}}$ are all isomorphic.

Definition 4. *Let $X \xrightarrow{f} Y \xleftarrow{f'} X'$ be two resolutions of a variety Y. Then this diagram is called a locally trivial family of Mukai flops of type A (resp. of type D) if there is a partial open covering $\{U_\lambda\}$ of Y which contains the singular locus of Y such that each diagram*

$$f^{-1}(U_\lambda) \to U_\lambda \leftarrow (f')^{-1}(U_\lambda)$$

is isomorphic to the product of a Mukai flop of type A (resp. of type D) with a suitable disc Δ^m.

4 Symplectic resolutions of Nilpotent orbits

In this section, we shall prove that any two symplectic resolutions of a nilpotent orbit closure of a classical simple Lie algebras are connected by a finite sequence of diagrams which are locally trivial families of Mukai flops.

Lemma 4.1. *Let $x \in \mathfrak{sl}(n)$ be a nilpotent element of type $\mathbf{d} := [d_1, ..., d_k]$. Let $(p_1, ..., p_s)$ be a sequence of positive integers such that $\mathrm{ord}(p_1, ..., p_s) = \mathbf{d}$. Fix a flag $F := \{F_i\}$ of $V := \mathbf{C}^n$ of type $(p_1, ..., p_s)$ such that $xF_i \subset F_{i-1}$ for all i. Assume that $p_j \neq p_{j-1}$ for an index j. Then we obtain a new flag F' of type $(p_1, ..., p_j, p_{j-1}, ..., p_s)$ from F such that $xF'_i \subset F'_{i-1}$ for all i by the following operation.*

(The case where $p_{j-1} < p_j$): x induces an endomorphism $\bar{x} \in \mathrm{End}(F_j/F_{j-2})$. For the projection $\phi : F_j \to F_j/F_{j-2}$, we put $F'_{j-1} := \phi^{-1}(\mathrm{Ker}(\bar{x}))$. We then put

$$F'_i := \begin{cases} F_i & (i \neq j-1) \\ F'_{j-1} & (i = j-1) \end{cases}$$

(The case where $p_{j-1} > p_j$): x induces an endomorphism $\bar{x} \in \mathrm{End}(F_j/F_{j-2})$. For the projection $\phi : F_j \to F_j/F_{j-2}$, we put $F'_{j-1} := \phi^{-1}(\mathrm{Im}(\bar{x}))$. We then put

$$F'_i := \begin{cases} F_i & (i \neq j-1) \\ F'_{j-1} & (i = j-1) \end{cases}$$

Lemma 4.2. *Let V be a $\mathbf{C}$-vector space of $\dim n$ with a non-degenerate bilinear form such that $< v, w >= (-1)^\epsilon < w, v >$ for all $v, w \in V$. Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{so}(V)$ or $\mathfrak{sp}(V)$ according as $\epsilon = 0$ or $\epsilon = 1$. Let $x \in \mathfrak{g}$ be a nilpotent element of type $\mathbf{d}$. Suppose that for $\pi \in \mathrm{Pai}(n, q)$, $\mathbf{d} = S(\pi)$ where S is the Spaltenstein map. Let $(p_1, ..., p_k, q, p_k, ..., p_1)$ be a sequence of integers such that $\pi = \mathrm{ord}(p_1, ..., p_k, q, p_k, ..., p_1)$. Fix an admissible flag F of type $(p_1, ..., p_k, q, p_k, ..., p_1)$ such that $xF_i \subset F_{i-1}$ for all i.*

(i) Assume that $p_{j-1} \neq p_j$ for an index $1 \leq j \leq k$. Then we obtain a new flag F' of type $(p_1, ..., p_j, p_{j-1}, ..., p_k, q, p_k, ..., p_{j-1}, p_j, ..., p_1)$ from F such that $xF'_i \subset F'_{i-1}$ for all i by the following operation.

(The case where $p_{j-1} < p_j$): x induces an endomorphism $\bar{x} \in \mathrm{End}(F_j/F_{j-2})$. For the projection $\phi : F_j \to F_j/F_{j-2}$, we put $F'_{j-1} := \phi^{-1}(\mathrm{Ker}(\bar{x}))$. We then put

$$F'_i := \begin{cases} F_i & (i \neq j-1, 2k+2-j) \\ F'_{j-1} & (i = j-1) \\ (F'_{j-1})^\perp & (i = 2k+2-j) \end{cases}$$

(The case where $p_{j-1} > p_j$): x induces an endomorphism $\bar{x} \in \mathrm{End}(F_j/F_{j-2})$. For the projection $\phi : F_j \to F_j/F_{j-2}$, we put $F'_{j-1} := \phi^{-1}(\mathrm{Im}(\bar{x}))$. We then put

$$F'_i := \begin{cases} F_i & (i \neq j-1, 2k+2-j) \\ F'_{j-1} & (i = j-1) \\ (F'_{j-1})^\perp & (i = 2k+2-j) \end{cases}$$

(ii) Assume that $q = 0$ and p_k is odd. Then there is an admissible flag F' of V of type $(p_1, ..., p_k, p_k, ..., p_1)$ such that

$xF'_i \subset F'_{i-1}$ for all i,

$F'_i = F_i$ for $i \neq k$ and

$F'_k \neq F_k$.

Proof. (i): When $p_{j-1} < p_j$, $\mathrm{rank}(\bar{x}) = p_{j-1}$ for $\bar{x} \in \mathrm{End}(F_j/F_{j-2})$. In fact, since $xF_j \subset F_{j-1}$, $\mathrm{rank}(\bar{x}) \leq p_{j-1}$. Assume that $\mathrm{rank}(\bar{x}) < p_{j-1}$. Then

we can construct a new flag from F by replacing F_{j-1} with a subspace F'_{j-1} containing F_{j-2} such that

$$\mathrm{Im}(\bar{x}) \subset F'_{j-1}/F_{j-2} \subset \mathrm{Ker}(\bar{x})$$

and $\dim F'_{j-1}/F_{j-2} = p_{j-1}$. The new flag satisfies $xF'_i \subset F'_{i-1}$ for all i and it has the same flag type as F. Since there are infinitely many choices of F'_{j-1}, we have infinitely many such F'. This contradicts the fact that x has only finite polarizations. Hence, $\mathrm{rank}(\bar{x}) = p_{j-1}$. Then the flag F' in our Lemma satisfies the desired properties. When $p_{j-1} > p_j$, we see that $\dim \mathrm{Ker}(\bar{x}) = p_{j-1}$ by a similar way. Then the latter argument is the same as when $p_{j-1} < p_j$.

(ii): According to the proof of Theorem 2.7. we construct a flag F such that $xF_i \subset F_{i-1}$. Since $q = 0$ and p_k is odd, we have the case (A) in the last step. As a consequence, we have two choices of the flags. One of them is F and another one is F'. Q.E.D.

Let F be the flag in Lemma 4.1, Lemma 4.2, (i) or Lemma 4.2, (ii). In each lemma, we have constructed another flag F'. Let G be the complex Lie group $SL(V)$, $Sp(V)$ or $SO(V)$ according as V is simply a $\mathbf{C}$-vector space with no bilinear forms, with a non-degenerate skew-symmetric form or with a non-degenerate symmetric form. Let $P \subset G$ (resp. $P' \subset G$) be the stabilizer group of the flag F (resp. F'). Then P and P' are both polarizations of $x \in \mathfrak{g}$. Let $\mathcal{O} \subset \mathfrak{g}$ be the nilpotent orbit containing x. Let us consider two Springer resolutions

$$T^*(G/P) \xrightarrow{\mu} \bar{\mathcal{O}} \xleftarrow{\mu'} T^*(G/P').$$

Note that $T^*(G/P)$ (resp. $T^*(G/P')$) is embedded in $G/P \times \bar{\mathcal{O}}$ (resp. $G/P' \times \bar{\mathcal{O}}$). An element $y \in \mathcal{O}$ can be written as $y = gxg^{-1}$ for some $g \in G$. Then we have

$$\mu^{-1}(y) = ([gF], y) \in G/P \times \bar{\mathcal{O}},$$

and

$$(\mu')^{-1}(y) = ([gF'], y) \in G/P' \times \bar{\mathcal{O}}.$$

We define the flag $\bar{F}$ in the following manner. If F is the flag in Lemma 4.1, then $\bar{F}$ is the flag obtained from F by deleting the subspace F_{j-1}. If F is the flag in Lemma 4.2, (i), then $\bar{F}$ is the flag obtained from F by deleting subspaces F_{j-1} and F_{2k+2-j}. Finally, if F is the flag in Lemma 4.2, (ii), then

$\bar{F}$ is the flag obtained from F by deleting F_k. Note that $\bar{F}$ is also obtained from F' by the same manner. Let $\bar{P} \subset G$ be the stabilizer group of the flag $\bar{F}$. We then have two projections

$$G/P \stackrel{p}{\to} G/\bar{P} \stackrel{p'}{\leftarrow} G/P'.$$

By two projections

$$G/P \times \bar{\mathcal{O}} \stackrel{p\times id}{\to} G/\bar{P} \times \bar{\mathcal{O}} \stackrel{p'\times id}{\leftarrow} G/P' \times \bar{\mathcal{O}},$$

$T^*(G/P)$ and $T^*(G/P')$ have the same image X in $G/\bar{P} \times \bar{\mathcal{O}}$. Since p and p' are both proper maps, X is a closed subvariety of $G/\bar{P} \times \bar{\mathcal{O}}$. The following diagram has been obtained as a consequence:

$$T^*(G/P) \to X \leftarrow T^*(G/P').$$

Lemma 4.3. *When F is the flag in Lemma 4.1 or in Lemma 4.2, (i), the diagram*

$$T^*(G/P) \stackrel{f}{\to} X \stackrel{f'}{\leftarrow} T^*(G/P')$$

is locally a trivial family of Mukai flops of type A. When F is the flag in Lemma 4.2, (ii), the diagram is locally a trivial family of Mukai flops of type D.

Proof. We prove the assetion when $\mathfrak{g} = \mathfrak{so}(n)$ or $\mathfrak{g} = \mathfrak{sp}(n)$. The case when $\mathfrak{g} = \mathfrak{sl}(n)$ is easier; so we omit the proof.

Consider the situation in Lemma 4.2, (i). A point of $G/\bar{P}$ corresponds to an isotropic flag $\bar{F}$ of V of type $(p_1, ..., p_{j-1}+p_j, ..., p_k, q, p_k, ..., p_{j-1}+p_j, ..., p_1)$. Let

$$0 \subset \bar{\mathcal{F}}_1 \subset ... \subset \bar{\mathcal{F}}_{2k-1} = (\mathcal{O}_{G/\bar{P}})^n$$

be the universal subundles on $G/\bar{P}$. Let

$$W \subset \underline{\mathrm{End}}(\bar{\mathcal{F}}_{j-1}/\bar{\mathcal{F}}_{j-2})$$

be the subvariety consisting of the points $([\bar{F}], \bar{x})$ where $\bar{x} \in \mathrm{End}(\bar{F}_{j-1}/\bar{F}_{j-2})$, $\bar{x}^2 = 0$ and $\mathrm{rank}(\bar{x}) \leq \min(p_j, p_{j-1})$. If we put $m := p_{j-1} + p_j$ and $r := \min(p_j, p_{j-1})$, then

$$W \to G/\bar{P}$$

is an $\bar{\mathcal{O}}_{[2^r,1^{m-2r}]}$ bundle over $G/\bar{P}$. Let us recall the definition of X.

$$X \subset G/\bar{P} \times \bar{\mathcal{O}}$$

consists of the points $([\bar{F}], x)$ such that $x\bar{F}_i \subset \bar{F}_{i-1}$ for all $i \neq j-1, 2k-j$ and $x\bar{F}_i \subset \bar{F}_i$ for $i = j-1, 2k-j$. Moreover, the induced endomorphism $\bar{x} \in \mathrm{End}(\bar{F}_{j-1}/\bar{F}_{j-2})$ satisfies $\bar{x}^2 = 0$ and $\mathrm{rank}(\bar{x}) \leq \min(p_{j-1}, p_j)$. Let

$$\phi : X \to W$$

be the projection defined by $\phi([\bar{F}], x) = ([\bar{F}], \bar{x})$, where $\bar{x} \in \mathrm{End}(\bar{F}_{j-1}/\bar{F}_{j-2})$ is the induced endomorphism by x. It can be checked that ϕ is an affine bundle. Since W is an $\bar{\mathcal{O}}_{[2^r,1^{m-2r}]}$ bundle over $G/\bar{P}$, there exists a family of Mukai flops of type A:

$$Y \to W \leftarrow Y'$$

parametrized by $G/\bar{P}$. The diagram

$$T^*(G/P) \to X \leftarrow T^*(G/P')$$

coincides with the pull back of the previous diagram by $\phi : X \to W$. Since ϕ is an affine bundle, this diagram is locally a trivial family of Mukai flops of type A.

Next consider the situation in Lemma 4.2, (ii). A point of $G/\bar{P}$ corresponds to an isotropic flag $\bar{F}$ of V of type $(p_1, ..., 2p_k, ..., p_1)$. Let

$$0 \subset \bar{\mathcal{F}}_1 \subset ... \subset \bar{\mathcal{F}}_{2k-1} = (\mathcal{O}_{G/\bar{P}})^n$$

be the universal subundles on $G/\bar{P}$. Let

$$W \subset \underline{\mathrm{End}}(\bar{\mathcal{F}}_k/\bar{\mathcal{F}}_{k-1})$$

be the subvariety consisting of the points $([\bar{F}], \bar{x})$ where

$$\bar{x} \in \bar{\mathcal{O}}_{[2^{p_k-1},1^2]} \subset \mathfrak{so}(\bar{F}_k/\bar{F}_{k-1}).$$

$W \to G/\bar{P}$ is an $\bar{\mathcal{O}}_{[2^{p_k-1},1^2]}$ bundle over $G/\bar{P}$. Let us recall the definition of X.

$$X \subset G/\bar{P} \times \bar{\mathcal{O}}$$

consists of the points $([\bar{F}], x)$ such that $x\bar{F}_i \subset \bar{F}_{i-1}$ for all $i \neq k$ and $x\bar{F}_k \subset \bar{F}_k$. Moreover, the induced endomorphism $\bar{x} \in \mathfrak{so}(\bar{F}_k/\bar{F}_{k-1})$ is contained in $\bar{\mathcal{O}}_{[2^{p_k-1},1^2]}$. Let

$$\phi : X \to W$$

be the projection defined by $\phi([\bar{F}], x) = ([\bar{F}], \bar{x})$, where $\bar{x} \in \mathfrak{so}(\bar{F}_k/\bar{F}_{k-1})$ is the induced endomorphism by x. It can be checked that ϕ is an affine bundle. Since W is an $\bar{\mathcal{O}}_{[2^{p_k-1},1^2]}$ bundle over $G/\bar{P}$, there exists a family of Mukai flops of type D:

$$Y \to W \leftarrow Y'$$

parametrized by $G/\bar{P}$. The diagram

$$T^*(G/P) \to X \leftarrow T^*(G/P')$$

coincides with the pull back of the previous diagram by $\phi : X \to W$. Since ϕ is an affine bundle, this diagram is locally a trivial family of Mukai flops of type D.

Theorem 4.4. *Let $\mathcal{O} \subset \mathfrak{g}$ be an orbit of a classical complex simple Lie algebra. Let Y and Y' be any two Springer resolutions of the closure $\bar{\mathcal{O}}$ of the nilpotent orbit. Then the birational map $Y - - \to Y'$ can be decomposed into finite number of diagrams $Y_i \to X_i \leftarrow Y_{i+1}$ with $Y_1 = Y$ and $Y_m = Y'$ in such a way that each diagram is locally a trivial family of Mukai flops of type A or of type D.*

Remark 4.5. *By a theorem of Fu [Fu 1, Thm 3.3], any projective symplectic resolution of $\bar{\mathcal{O}}$ is obtained as a Springer resolution.*

Proof. (The case $\mathfrak{g} = \mathfrak{sl}(V)$): We put $G = SL(V)$. Let $Y = T^*(G/P)$ and $Y' = T^*(G/P')$, where P and P' are polarizations of an element $x \in \mathcal{O}$. By Theorem 2.6, we may assume that P has the flag type $(p_1, ..., p_s)$ and P' has the flag type $(p_{\sigma(1)}, ..., p_{\sigma(s)})$ where σ is a permutation of $\{1, 2, ..., s\}$. Let F be the flag of V stabilized by P. Applying the operations in Lemma 4.1 successively, one can reach a flag F' of type $(p_{\sigma(1)}, ..., p_{\sigma(s)})$. Note that P' is the stabilizer group of F'. By Lemma 4.3, each step corresponds to a diagram which is locally a trivial family of a Mukai flop of type A.

(The case $\mathfrak{g} = \mathfrak{sp}(V)$ or $\mathfrak{so}(V)$): Let G be $SO(V)$ or $Sp(V)$. By Theorem 2.8, the parabolic subgroups giving Springer resolutions of $\bar{\mathcal{O}}$ all have the same Levi type $\pi \in \mathrm{Pai}(n, q)$.

Assume that $q \neq 0$ or $\epsilon = 1$. We write $\pi = \mathrm{ord}(p_1, ..., p_k, q, p_k, ..., p_1)$ with a non-decreasing sequence $p_1 \leq p_2 \leq ... \leq p_k$. Fix an element $x \in \mathcal{O}$. Let $Y = T^*(G/P)$ with a polarization P of x. The flag type of P is $(p_{\sigma(1)}, ..., p_{\sigma(k)}, q, p_{\sigma(k)}, ..., p_{\sigma(1)})$ with a permutation σ of $\{1, 2, ..., k\}$. Since

$q \neq 0$ or $\epsilon = 1$, the conjugacy class of a parabolic subgroups is uniquely determined by its flag type. Let P_0 be a polarization of x with the flag type $(p_1, ..., p_k, q, p_k, ..., p_1)$. Let F be an admissible flag of V stabilized by P. Applying the operations in Lemma 4.2, (i) successively, one can reach a flag F' of type $(p_1, ..., p_k, q, p_k, ..., p_1)$. Then P_0 is the stabilizer group of F'. By Lemma 4.3, each step corresponds to a diagram which is locally a trivial family of a Mukai flop of type A. Therefore, the birational map $T^*(G/P) -- \to T^*(G/P_0)$ is decomposed into desired diagrams. By the same argument, the birational map $T^*(G/P') -- \to T^*(G/P_0)$ is also decomposed into desired diagrams. This shows that Y and Y' are connected by diagrams which are locally trivial families of Mukai flops of type A.

Assume that $q = 0$ and $\epsilon = 0$. Since we are only concerned with parabolic subgroups whose Springer maps are birational, the members of ${}^t\pi$ (cf. Notation and Convention) are all even numbers or even if some odd numbers appear in ${}^t\pi$, they are all the same number (possibly with some multiplicity).

(a) *The case where all members of* ${}^t\pi$ *are even*:

We write $\pi = \mathrm{ord}(p_1, ..., p_k, p_k, ..., p_1)$ with a non-decreasing sequence $p_1 \leq p_2 \leq ... \leq p_k$. Our nilpotent orbit $\mathcal{O}$ is one of the nilpotent orbits with a *very even* partition. Fix an element $x \in \mathcal{O}$. Then a polarization of x is uniquely determined by its flag type. Let P_0 be a polarization of x with the flag type $(p_1, ..., p_k, p_k, ..., p_1)$. The symplectic resolution Y can be written as $T^*(G/P)$ with a polarization P of x. The flag type of P is $(p_{\sigma(1)}, ..., p_{\sigma(k)}, p_{\sigma(k)}, ..., p_{\sigma(1)})$ with some permutation σ of $\{1, 2, ..., k\}$. Let F be an admissible flag of V stabilized by P. Applying the operations in Lemma 4.2, (i) successively, one can reach a flag F' of type $(p_1, ..., p_k, p_k, ..., p_1)$. Then P_0 is the stabilizer group of F'. By Lemma 4.3, each step corresponds to a diagram which is locally a trivial family of a Mukai flop of type A. Therefore, the birational map $T^*(G/P) - - \to T^*(G/P_0)$ is decomposed into desired diagrams. On the other hand, another symplectic resolution Y' is written as $T^*(G/P')$ with a polarization P' of x. By the same argument, the birational map $T^*(G/P') - - \to T^*(G/P_0)$ is also decomposed into desired diagrams. This shows that Y and Y' are connected by diagrams which are locally trivial families of Mukai flops of type A.

(b) *The case where odd numbers appear in* ${}^t\pi$:

One can write $\pi = \mathrm{ord}(p_1, ..., p_k, p_k, ..., p_1)$ in such a way that $p_1 \leq ... \leq p_l$ are even numbers and that $p_{l+1} = ... = p_k$ are odd. Fix $x \in \mathcal{O}$, and

write $Y = T^*(G/P)$ and $Y' = T^*(G/P')$ with polarizations P and P' of x. Let $(p_{\sigma(1)}, ..., p_{\sigma(k)}, p_{\sigma(k)}, ..., p_{\sigma(1)})$ and $(p_{\tau(1)}, ..., p_{\tau(k)}, p_{\tau(k)}, ..., p_{\tau(1)})$ be the flag types of P and P' respectively. Let F (resp. F') be an admissible flag stabilized by P (resp. P'). Applying the operations in Lemma 4.2, (i) to F (resp. F') successively, one can reach a flag F^0 (resp. $(F')^0$) of type $(p_1, ..., p_k, p_k, ..., p_1)$. Let P_0 (resp. P'_0) be the stabilizer group of F^0 (resp. $(F')^0$). Then the birational maps $T^*(G/P) -- \to T^*(G/P_0)$ and $T^*(G/P') -- \to T^*(G/(P')_0)$ are decomposed into diagrams which are locally trivial families of Mukai flops of type A. If $p_{l+1} = ... = p_k = 1$, then P_0 and P'_0 are conjugate to each other. In this case, two Springer resolutions $T^*(G/P_0) \to \bar{\mathcal{O}}$ and $T^*(G/(P')_0) \to \bar{\mathcal{O}}$ are isomorphic. Then the birational map $T^*(G/P) -- \to T^*(G/P')$ is decomposed into diagrams which are locally trivial families of Mukai flops of type A. If $p_{l+1} = ... = p_k > 1$, then P_0 and P'_0 may not be conjugate. When P_0 and P'_0 are conjugate, the birational map $T^*(G/P) -- \to T^*(G/P')$ is decomposed into diagrams which are locally trivial families of Mukai flops of type A. When P_0 and P'_0 are not conjugate, the relationship between two flags F^0 and $(F')^0$ are described in Lemma 4.2, (ii). In this case, $T^*(G/P_0) -- \to T^*(G/P'_0)$ is locally a trivial family of Mukai flops of type D. Then the birational map $T^*(G/P) -- \to T^*(G/P')$ is decomposed into diagrams which are locally trivial families of Mukai flops of type A and of type D.

Example 4.6. *Let $\mathcal{O} \subset \mathfrak{sl}(6)$ be the nilpotent orbit of Jordan type $[3,2,1]$. Take an element $x \in \mathcal{O}$. Then x has six polarizations $P_{\sigma(1),\sigma(2),\sigma(3)} \subset SL(6)$ of flag types $(\sigma(1), \sigma(2), \sigma(3))$ where σ are permutations of $\{1,2,3\}$. Put $Y_{i,j,k} := T^*(SL(6)/P_{i,j,k})$. They are symplectic resolutions of $\bar{\mathcal{O}}$. Then we have a link of birational maps:*

$$-- \to Y_{321} -- \to Y_{231} -- \to Y_{213} -- \to$$

$$Y_{123} -- \to Y_{132} -- \to Y_{312} -- \to Y_{321}.$$

Each birational map fits into a diagram which is locally a trivial family of Mukai flops of type A. For example, Y_{321} and Y_{231} are linked by a diagram

$$Y_{321} \to X_{5,1} \leftarrow Y_{231}.$$

This diagram is locally a trivial family of the Mukai flop

$$T^*G(2,5) \to \bar{\mathcal{O}}_{[2^2,1]} \leftarrow T^*G(3,5).$$

Let $N^1(Y_{i,j,k})$ be the Abelian groups of $\bar{\mathcal{O}}$-numerical classes of $\mathbf{R}$-divisors on $Y_{i,j,k}$. Note that $N^1(Y_{i,j,k}) \cong \mathbf{R}^2$. Since $Y_{i,j,k}$ are isomorphic in codimension 1, $N^1(Y_{i,j,k})$ are naturally identified. Hence we denote by N^1 these $\mathbf{R}$-vector spaces. N^1 is divided into six chambers, each of which corresponds to the ample cone $\overline{\mathrm{Amp}}(Y_{i,j,k})$.

Example 4.7. *Let $\mathcal{O} \subset \mathfrak{so}(10)$ be the nilpotent orbit of Jordan type $[4^2, 1^2]$. Take an element x of $\mathcal{O}$. Then x has four polarizations $P^+_{3,2,2,3}$, $P^-_{3,2,2,3}$, $P^+_{2,3,3,2}$ and $P^-_{2,3,3,2}$, each of which has the flag type indicated by the index. Note that there are different polarizations which have the same flag type. We put $Y^+_{i,j,j,i} := T^*(SO(10)/P^+_{i,j,j,i})$ and $Y^-_{i,j,j,i} := T^*(SO(10)/P^-_{i,j,j,i})$. They are symplectic resolutions of $\bar{\mathcal{O}}$. Then we have a link of birational maps*

$$Y^+_{3,2,2,3} \dashrightarrow Y^+_{2,3,3,2} \dashrightarrow Y^-_{2,3,3,2} \dashrightarrow Y^-_{3,2,2,3}.$$

Here $Y^+_{3,2,2,3}$ and $Y^+_{2,3,3,2}$ are linked by a diagram

$$Y^+_{3,2,2,3} \to X^+_{5,5} \leftarrow Y^+_{2,3,3,2}.$$

This diagram is locally a trivial family of the Mukai flop

$$T^*G(3,5) \to \bar{\mathcal{O}}_{[2^2,1]} \leftarrow T^*G(2,5).$$

The same picture can be drawn for $Y^-_{3,2,2,3}$ and $Y^-_{2,3,3,2}$. Finally, $Y^+_{2,3,3,2}$ and $Y^-_{2,3,3,2}$ are linked by a diagram

$$Y^+_{2,3,3,2} \to X^+_{2,6,2} \leftarrow Y^-_{2,3,3,2}.$$

This diagram is locally a trivial family of the Mukai flop

$$T^*G^+_{iso}(3,6) \to \bar{\mathcal{O}}_{[2^2,1^2]} \leftarrow T^*G^-_{iso}(3,6).$$

$Y^+_{2,3,3,2}$ (resp. $Y^-_{2,3,3,2}$) has two small birational contraction maps. On the other hand, $Y^+_{3,2,2,3}$ (resp. $Y^-_{3,2,2,3}$) has a small birational contraction map and a divisorial birational contraction map. This is the reason why the link of birational maps above ends at $Y^+_{3,2,2,3}$ and $Y^-_{3,2,2,3}$. Let $N^1(Y^+_{i,j,j,i})$ (resp. $N^1(Y^-_{i,j,j,i})$) be the $\mathbf{R}$-vector space of $\bar{\mathcal{O}}$-numerical classes of $\mathbf{R}$-divisors on $Y^+_{i,j,j,i}$ (resp. $Y^-_{i,j,j,i}$). Note that these spaces have dimension 2. Since $Y^+_{i,j,j,i}$ and $Y^-_{i,j,j,i}$ are isomorphic in codimension 1, these $\mathbf{R}$-vector spaces are naturally identified. Hence we denote them by N^1. Let $\mathrm{Mov} \subset N^1$ be the subcone generated by movable $\mathbf{R}$ divisors. In our case Mov does not coincides with N^1. The closure $\overline{\mathrm{Mov}}$ of Mov is divided into four chambers, each of which corresponds to the ample cone $\overline{\mathrm{Amp}}(Y^+_{i,j,j,i})$ or $\overline{\mathrm{Amp}}(Y^-_{i,j,j,i})$.

Example 4.8. *Let $\mathcal{O} \subset \mathfrak{so}(10)$ be the nilpotent orbit of Jordan type $[3^2, 2^2]$. Take an element x of $\mathcal{O}$. Then x has three polarizations $P^+_{1,4,4,1}$, $P^-_{1,4,4,1}$ and $P_{4,1,1,4}$, each of which has the flag type indicated by the index. Note that there are different polarizations which have the same flag type. We put $Y^+_{1,4,4,1} := T^*(SO(10)/P^+_{1,4,4,1})$, $Y^-_{1,4,4,1} := T^*(SO(10)/P^-_{1,4,4,1})$ and $Y_{4,1,1,4} := T^*(SO(10)/P_{4,1,1,4})$. They are symplectic resolutions of $\bar{\mathcal{O}}$. Then we have a link of birational maps*

$$Y^+_{1,4,4,1} - - \to Y_{4,1,1,4} - - \to Y^-_{1,4,4,1}.$$

Here $Y^+_{1,4,4,1}$ and $Y_{4,1,1,4}$ are linked by a diagram

$$Y^+_{1,4,4,1} \to X^+_{5,5} \leftarrow Y_{4,1,1,4}.$$

This diagram is locally a trivial family of the Mukai flop

$$T^*G(1,5) \to \bar{\mathcal{O}}_{[2,1^3]} \leftarrow T^*G(4,5).$$

The same picture can be drawn for $Y^-_{1,4,4,1}$ and $Y_{4,1,1,4}$. $Y_{4,1,1,4}$ has two small birational contraction maps. On the other hand, $Y^+_{1,4,4,1}$ (resp. $Y^-_{1,4,4,1}$) has a small birational contraction map and a divisorial birational contraction map. This is the reason why the link of birational maps above ends at $Y^+_{1,4,4,1}$ and $Y^-_{1,4,4,1}$. Let $N^1(Y^+_{1,4,4,1})$ (resp. $N^1(Y^-_{1,4,4,1})$, $N^1(Y_{4,1,1,4})$) be the $\mathbf{R}$-vector space of $\bar{\mathcal{O}}$-numerical classes of $\mathbf{R}$ divisors of $Y^+_{1,4,4,1}$ (resp. $Y^-_{1,4,4,1}$, $Y_{4,1,1,4}$). Since $Y^+_{1,4,4,1}$, $Y^-_{1,4,4,1}$ and $Y_{4,1,1,4}$ are isomorphic in codimension 1, these $\mathbf{R}$-vector spaces are naturally identified. Hence we denote them by N^1. Let $\mathrm{Mov} \subset N^1$ be the subcone generated by movable $\mathbf{R}$ divisors. In our case Mov does not coincides with N^1. The closure $\overline{\mathrm{Mov}}$ of Mov is divided into three chambers. They correspond to the ample cones of three resolutions of $\bar{\mathcal{O}}$.

The following is a special case of a more general problem (cf. [Na 2]).

Conjecture 1. *All symplectic resolutions of a nilpotent orbit closure in a classical Lie algebra have equivalent bounded derived categories of coherent sheaves.*

In the conjecture, the equivalences between derived categories should respect the birational map between resolutions. More explicitly, let Y and Y' be two symplectic resolutions and let $U \subset Y$ and $U' \subset Y'$ be Zariski open subsets such that U is isomorphically mapped onto U' by the natural birational

map $\phi : Y \to Y'$. Then we want to have an equivalece $F : D(Y) \to D(Y')$ such that $F(\mathcal{O}_y) \cong \mathcal{O}_{\phi(y)}$ for all $y \in U$, where $\mathcal{O}_y$ (resp. $\mathcal{O}_{\phi(y)}$) is the structure sheaf of the closed point y (resp. $\phi(y)$). By Theorem 4.4 and by [Na 1], §5, the conjecture is reduced to the cases of Mukai flops of type A and of type D.

5 Deformations of nilpotent orbits

Let $x \in \mathfrak{g}$ be a nilpotent element of a Lie algebra attached to a classical simple complex Lie group G. Let $\mathcal{O}$ be the nilpotent orbit containing x. In this section, by using an idea of Borho and Kraft [B-K], we shall construct a *deformation*[1] $f : \mathcal{S} \to \mathfrak{k}$ of $\bar{\mathcal{O}}$ such that

(i) $f^{-1}(0) = \bar{\mathcal{O}}$ for $0 \in \mathfrak{k}$, and

(ii) for any Springer resolution $T^*(G/P) \to \bar{\mathcal{O}}$, there is a *simultaneous resolution*

$$\tau_P : \mathcal{E}_P \to \mathfrak{k}$$

of f, where $(\tau_P)^{-1}(0) = T^*(G/P)$ and where $(\tau_P)^{-1}(t) \cong f^{-1}(t)$ for a general point $t \in \mathfrak{k}$.

As a corollary of this construction, we can verify Conjecture 2 in [F-N] for the closure of a nilpotent orbit of a classical simple Lie algebra. Conjecture 2 has already been proved for $\mathfrak{sl}(n)$ in [F-N], Theorem 4.4 in a very explicit form. Note that, a weaker version of this conjecture has been proved by Fu [Fu 2] for the closure of a nilpotent orbit of a classical simple Lie algebra.

$\mathfrak{g}$ becomes a G-variety via the adjoint action. Let $Z \subset \mathfrak{g}$ be a closed subvariety. For $m \in \mathbf{N}$, put

$$Z^{(m)} := \{x \in Z; \dim Gx = m\}.$$

$Z^{(m)}$ becomes a locally closed subset of Z. We put $m(Z) := \max\{m; m = \dim Gx, \exists x \in Z\}$. Then $Z^{m(Z)}$ is an open subset of Z, which will be denoted by Z^{reg}. A *sheet* of Z is an irreducible component of some $Z^{(m)}$. A sheet of $\mathfrak{g}$ is called a *Dixmier sheet* if it contains a semi-simple element of $\mathfrak{g}$.

Let $P \subset G$ be a parabolic subgroup and let $\mathfrak{p}$ be its Lie algebra. Let $\mathfrak{m}(P)$ be the Levi factor of $\mathfrak{p}$. We put $\mathfrak{k}(P) := \mathfrak{g}^{\mathfrak{m}(P)}$ where

$$\mathfrak{g}^{\mathfrak{m}(P)} := \{x \in \mathfrak{g}; [x, y] = 0, \forall y \in \mathfrak{m}(P)\}.$$

[1]Here we do not assume that f is flat, but only assume that all fibers of f has the same dimension. This suffices for an application.

Let $\mathfrak{r}(P)$ be the radical of $\mathfrak{p}$.

Theorem 5.1. *$G\mathfrak{r}(P) = \overline{G\mathfrak{k}(P)}$ and $G\mathfrak{r}(P)^{\mathrm{reg}}$ $(= \overline{G\mathfrak{k}(P)}^{\mathrm{reg}})$ is a Dixmier sheet.*

Proof. See [B-K], Satz 5.6.

Every element x of $\mathfrak{g}$ can be uniquely written as $x = x_n + x_s$ with x_n nilpotent and with x_s semi-simple such that $[x_n, x_s] = 0$. Let $\mathfrak{h}$ be a Cartan subalgebra and let W be the Weyl group with respect to $\mathfrak{h}$. The set of semi-simple orbits is identified with $\mathfrak{h}/W$. Let $\mathfrak{g} \to \mathfrak{h}/W$ be the map defined as $x \to [\mathcal{O}_{x_s}]$. There is a direct sum decomposition

$$\mathfrak{r}(P) = \mathfrak{k}(P) \oplus \mathfrak{n}(P), (x \to x_1 + x_2)$$

where $\mathfrak{n}(P)$ is the nil-radical of $\mathfrak{p}$ (cf. [Slo], 4.3). We have a well-defined map

$$G \times_P \mathfrak{r}(P) \to \mathfrak{k}(P)$$

by sending $[g, x] \in G \times_P \mathfrak{r}(P)$ to $x_1 \in \mathfrak{k}(P)$ and there is a commutative diagram

$$\begin{array}{ccc} G \times_P \mathfrak{r}(P) & \to & G\mathfrak{r}(P) \\ \downarrow & & \downarrow \\ \mathfrak{k}(P) & \to & \mathfrak{h}/W. \end{array}$$

by [Slo], 4.3.

Lemma 5.2. *The induced map*

$$G \times_P \mathfrak{r}(P) \xrightarrow{\mu_P} \mathfrak{k}(P) \times_{\mathfrak{h}/W} G\mathfrak{r}(P)$$

is a birational map.

Proof. Let $h \in \mathfrak{k}(P)^{\mathrm{reg}}$ and denote by $\bar{h} \in \mathfrak{h}/W$ its image by the map $\mathfrak{k}(P) \to \mathfrak{h}/W$. Then the fiber of the map $G\mathfrak{r}(P) \to \mathfrak{h}/W$ over $\bar{h}$ coincides with a semi-simple orbit $\mathcal{O}_h$ of $\mathfrak{g}$ containing h. In fact, by Theorem 5.1, the fiber actally contains this orbit. The fiber is closed in $\mathfrak{g}$ because $G\mathfrak{r}(P)$ is closed subset of $\mathfrak{g}$ by Theorem 5.1. Note that a semi-simple orbit of $\mathfrak{g}$ is also closed. Hence if the fiber and $\mathcal{O}_h$ does not coincide, then the fiber contains an orbit with larger dimension than $\dim \mathcal{O}_h$. This contradicts the fact that $\overline{G\mathfrak{k}(P)^{\mathrm{reg}}} = G\mathfrak{r}(P)$. Take a point $(h, h') \in \mathfrak{k}(P)^{\mathrm{reg}} \times_{\mathfrak{h}/W} G\mathfrak{r}(P)$. Then h' is

a semi-simple element G-conjugate to h. Fix an element $g_0 \in G$ such that $h' = g_0 h (g_0)^{-1}$. We have

$$(\mu_P)^{-1}(h, h') = \{[g, x] \in G \times_P \mathfrak{r}(P);\, x_1 = h,\, gxg^{-1} = h'\}.$$

Since $x = px_1p^{-1}$ for some $p \in P$ and conversely $(px_1p^{-1})_1 = x_1$ for any $p \in P$ (cf. [Slo], Lemma 2, p.48), we have

$$(\mu_P)^{-1}(h, h') = \{[g, php^{-1}] \in G \times_P \mathfrak{r}(P);\, g \in G,\, p \in P,\, (gp)h(gp)^{-1} = h'\} =$$

$$\{[gp, h] \in G \times_P \mathfrak{r}(P);\, g \in G,\, p \in P,\, (gp)h(gp)^{-1} = h'\} =$$

$$\{[g, h] \in G \times_P \mathfrak{r}(P);\, ghg^{-1} = h'\} =$$

$$\{[g_0 g', h] \in G \times_P \mathfrak{r}(P);\, g' \in Z_G(h)\} = g_0(Z_G(h)/Z_P(h)).$$

Here $Z_G(h)$ (resp. $Z_P(h)$) is the centralizer of h in G (resp. P). By [Ko], 3.2, Lemma 5, $Z_G(h)$ is connected. Moreover, since $\mathfrak{g}^h \subset \mathfrak{p}$, $\mathrm{Lie}(Z_G(h)) = \mathrm{Lie}(Z_P(h))$. Therefore, $Z_G(h)/Z_P(h) = \{1\}$, and $(\mu_P)^{-1}(h, h')$ consists of one point.

Lemma 5.3. *The map*

$$G \times_P \mathfrak{r}(P) \to G\mathfrak{r}(P)$$

is a proper map.

Proof. As a vector subbundle, we have a closed immersion

$$G \times_P \mathfrak{r}(P) \to G/P \times \mathfrak{g}.$$

This map factors through $G/P \times G\mathfrak{r}(P)$, and hence we have a closed immersion

$$G \times_P \mathfrak{r}(P) \to G/P \times G\mathfrak{r}(P).$$

Our map is the composition of this closed immersion and the projection

$$G/P \times G\mathfrak{r}(P) \to G\mathfrak{r}(P).$$

Since G/P is compact, this projection is a proper map.

Lemma 5.4. *Let $x \in \mathfrak{g}$ be a nilpotent orbit of a classical simple Lie algebra and denote by $\mathcal{O}$ the nilpotent orbit containing x. Then the polarizations of x giving Springer resolutions of $\bar{\mathcal{O}}$ all have conjugate Levi factors.*

Proof. By Theorem 2.8, the Levi type π of such a polarization P is unique. Parabolic subgroups of G with the same Levi type have congugate Levi factors except in the case $q = \epsilon = 0$ and $\pi^i \equiv 0 \pmod 2$ $\forall i$ (cf. [He], Lemma 4.6,(c)). In this exceptional case, there are two conjugacy classes of parabolic subgroups having non-conjugate Levi factors. They have mutually different Richardson orbits $\mathcal{O}^I$ and $\mathcal{O}^{II}$. Here $\mathcal{O}^I$ and $\mathcal{O}^{II}$ are very even orbits with the same Jordan type. In particular, it is impossible that two parabolic subgroups having non-conjugate Levi factors becomes polarizations of the same element x.

Lemma 5.5. *Let $x \in \mathfrak{g}$ be the same as the previous lemma. Let P and P' be polarizations of x. Assume that they both give Springer resolutions of $\bar{\mathcal{O}}$. Then $\mathfrak{k}(P)$ and $\mathfrak{k}(P')$ are conjugate to each other.*

Proof. Let M_P and $M_{P'}$ be Levi factors of P and P' respectively. Then M_P and $M_{P'}$ are conjugate by the previous lemma. Hence their centralizers are also conjugate. The Lie algebras of these centralizers are $\mathfrak{k}(P)$ and $\mathfrak{k}(P')$.

Corollary 5.6. *For P, $P' \in \mathrm{Pol}(x)$ which give Springer resolutions of $\bar{\mathcal{O}}$, we have $G\mathfrak{r}(P) = G\mathfrak{r}(P')$.*

Proof. By Theorem 5.1, $G\mathfrak{r}(P) = \overline{G\mathfrak{k}(P)}$ and $G\mathfrak{r}(P') = \overline{G\mathfrak{k}(P')}$. Since $G\mathfrak{k}(P) = G\mathfrak{k}(P')$, we have the result.

Lemma 5.7. *The image of the composed map*

$$G\mathfrak{r}(P) \to \mathfrak{g} \to \mathfrak{h}/W$$

coincides with $\mathfrak{k}(P)/W_P$, where

$$W_P = \{w \in W; w(\mathfrak{k}(P)) = \mathfrak{k}(P)\}.$$

Proof. By definition, $\mathfrak{k}(P)/W_P \subset \mathfrak{h}/W$, which is a closed subset. Since $G\mathfrak{r}(P) = \overline{G\mathfrak{k}(P)}$, we only have to prove that the image of $G\mathfrak{k}(P)$ by the map $\mathfrak{g} \to \mathfrak{h}/W$ coincides with $\mathfrak{k}(P)/W_P$. Every element of $G\mathfrak{k}(P)$ is semi-simple, and the map $G\mathfrak{k}(P) \to \mathfrak{h}/W$ sends an element of $G\mathfrak{k}(P)$ to its (semi-simple) orbit. Hence the image coincides with $\mathfrak{k}(P)/W_P$.

Corollary 5.8. $\mathfrak{k}(P)$ *and* $\mathfrak{k}(P')$ *are W-conjugate in $\mathfrak{h}$.*

Proof. Let $q : \mathfrak{h} \to \mathfrak{h}/W$ be the quotient map. Since $G\mathfrak{r}(P) = G\mathfrak{r}(P')$, $q(\mathfrak{k}(P)) = q(\mathfrak{k}(P'))$ by the previous lemma. Put $\mathfrak{k}_\pi := q(\mathfrak{k}(P))$. Then $\mathfrak{k}(P)$ and $\mathfrak{k}(P')$ are both irreducible components of $q^{-1}(\mathfrak{k}_\pi)$. Hence, $\mathfrak{k}(P)$ and $\mathfrak{k}(P')$ are W-conjugate in $\mathfrak{h}$.

We fix a polarization P_0 of x which gives a Springer resolution of $\bar{\mathcal{O}}$. Let P be another such polarization. By the corollary above, $\mathfrak{k}(P)$ and $\mathfrak{k}(P_0)$ are finite coverings of $\mathfrak{k}_\pi$, and there is a $\mathfrak{k}_\pi$-isomorphism $\mathfrak{k}(P) \cong \mathfrak{k}(P_0)$. We fix such an isomorphism. Then it induces an isomorphism

$$\mathfrak{k}(P) \times_{\mathfrak{h}/W} G\mathfrak{r}(P) \xrightarrow{\iota_P} \mathfrak{k}(P_0) \times_{\mathfrak{h}/W} G\mathfrak{r}(P_0).$$

We put $\nu_P := \iota_P \circ \mu_P$, and

$$\mathcal{S} := \mathfrak{k}(P_0) \times_{\mathfrak{h}/W} G\mathfrak{r}(P_0).$$

Denote by f the first projection $\mathcal{S} \to \mathfrak{k}(P_0)$. Then $f^{-1}(0) = \bar{\mathcal{O}}$, and for each polarization P of x,

$$G \times_P \mathfrak{r}(P) \xrightarrow{\nu_P} \mathcal{S} \to \mathfrak{k}(P_0)$$

gives a simultaneous resolution of f. This simultaneous resolution coincides with the Springer resolution $T^*(G/P) \to \bar{\mathcal{O}}$ over $0 \in \mathfrak{k}(P_0)$.

The following conjecture is posed in [F-N].

Conjecture 2. *Let Z be a normal symplectic singularity. Then for any two symplectic resolutions $f_i : X_i \to Z$, $i = 1, 2$, there are flat deformations $\mathcal{X}_i \xrightarrow{F_i} \mathcal{Z} \to T$ such that, for $t \in T - \{0\}$, $F_{i,t} : \mathcal{X}_{i,t} \to \mathcal{Z}_t$ are isomorphisms.*

Theorem 5.9. *The conjecture holds for the normalization $\tilde{\mathcal{O}}$ of a nilpotent orbit closure $\bar{\mathcal{O}}$ in a classical Lie algebra.*

Proof. By [Fu 1], Theorem 3.3, all symplectic resolutions of $\tilde{\mathcal{O}}$ are Springer resolutions. Take a general curve $T \subset \mathfrak{k}(P_0)$ passing through $0 \in \mathfrak{k}(P_0)$, and pull back the family

$$G \times_P \mathfrak{r}(P) \xrightarrow{\nu_P} \mathcal{S} \to \mathfrak{k}(P_0)$$

by $T \to \mathfrak{k}(P_0)$. Put $\bar{\mathcal{Z}} := \mathcal{S} \times_{\mathfrak{k}(P_0)} T$. Then, for each P, we have a simultaneous resolution of $\bar{\mathcal{Z}} \to T$:

$$\mathcal{X}_P \to \bar{\mathcal{Z}} \to T.$$

Let $\mathcal{Z}$ be the normalization of $\bar{\mathcal{Z}}$. Then the map $\mathcal{X}_P \to \bar{\mathcal{Z}}$ factors through $\mathcal{Z}$. Now

$$\mathcal{X}_P \to \mathcal{Z} \to T$$

gives a desired deformation of the Springer resolution $T^*(G/P) \to \tilde{\mathcal{O}}$.

Example 5.10. *Our abstract construction coincides with the explicit construction in [F-N], Theorem 4.4 in the case where $\mathfrak{g} = \mathfrak{sl}(n)$. Let us briefly observe the correspondence between two constructions. Assume that $\mathcal{O}_x \subset \mathfrak{sl}(n)$ is the orbit containing an nilpotent element x of type $\mathbf{d} := [d_1, ..., d_k]$. Let $[s_1, ..., s_m]$ be the dual partition of $\mathbf{d}$ (cf. Notation and Convention). By Theorem 2.6, the polarizations of x have the flag type $(s_{\sigma(1)}, \cdots, s_{\sigma(m)})$ with $\sigma \in \Sigma_m$. We denote them by P_σ. We put $P_0 := P_{id}$. Define $F_\sigma := SL(n)/P_\sigma$. Let*

$$\tau_1 \subset \cdots \subset \tau_{m-1} \subset \mathbb{C}^n \otimes_{\mathbb{C}} \mathcal{O}_{F_\sigma}$$

be the universal subbundles on F_σ. A point of T^*F_σ is expressed as a pair (p, ϕ) of $p \in F_\sigma$ and $\phi \in \mathrm{End}(\mathbb{C}^n)$ such that

$$\phi(\mathbb{C}^n) \subset \tau_{m-1}(p), \cdots, \phi(\tau_2(p)) \subset \tau_1(p), \phi(\tau_1(p)) = 0.$$

The Springer resolution

$$s_\sigma : T^*F_\sigma \to \bar{\mathcal{O}}$$

is defined as $s_\sigma((p, \phi)) := \phi$. In [F-N], Theorem 4.4, we have next defined a vector bundle $\mathcal{E}_\sigma$ with an exact sequence

$$0 \to T^*F_\sigma \to \mathcal{E}_\sigma \stackrel{\eta_\sigma}{\to} \mathcal{O}_{F_\sigma}^{m-1} \to 0.$$

For $p \in F_\sigma$, we can choose a basis of $\mathbf{C}^n$ such that $T^*F_\sigma(p)$ consists of the matrices of the following form

$$\begin{pmatrix} 0 & * & \cdots & * \\ 0 & 0 & \cdots & * \\ \cdots & & & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then $\mathcal{E}_\sigma(p)$ is the vector subspace of $\mathfrak{sl}(n)$ consisting of the matrices A of the following form

$$\begin{pmatrix} a_{\sigma(1)} & * & \cdots & * \\ 0 & a_{\sigma(2)} & \cdots & * \\ \cdots & & & \cdots \\ 0 & 0 & \cdots & a_{\sigma(m)} \end{pmatrix},$$

where $a_i := a_i I_{s_i}$ and I_{s_i} is the identity matrix of the size $s_i \times s_i$. Since $A \in \mathfrak{sl}(n)$, $\Sigma_i s_i a_i = 0$. Here we define the map $\eta_\sigma(p) : \mathcal{E}_\sigma(p) \to \mathbb{C}^{\oplus m-1}$

as $\eta_\sigma(p)(A) := (a_1, a_2, \cdots, a_{m-1})$*. This vector bundle* $\mathcal{E}_\sigma$ *is nothing but our* $SL(n) \times_{P_\sigma} \mathfrak{r}(P_\sigma)$*. Moreover, the map*

$$\eta_\sigma : \mathcal{E}_\sigma \to \mathbb{C}^{m-1}$$

coincides with the map

$$SL(n) \times_{P_\sigma} \mathfrak{r}(P_\sigma) \to \mathfrak{k}(P_0),$$

where we identify $\mathfrak{k}(P_\sigma)$ *with* $\mathfrak{k}(P_0)$ *by an* $\mathfrak{k}_\pi$*-isomorphism. Finally, in [F-N], Theorem 4.4 we have defined* $\overline{N} \subset \mathfrak{sl}(n)$ *to be the set of all matrices which is conjugate to a matrix of the following form:*

$$\begin{pmatrix} b_1 & * & \cdots & * \\ 0 & b_2 & \cdots & * \\ \cdots & & & \cdots \\ 0 & 0 & \cdots & b_m \end{pmatrix},$$

where $b_i = b_i I_{s_i}$ *and* I_{s_i} *is the identity matrix of order* s_i*. Furthermore the zero trace condition* $\sum_i s_i b_i = 0$ *was required. For* $A \in \overline{N}$*, let* $\phi_A(x) := \det(xI - A)$ *be the characteristic polynomial of* A*. Let* $\phi_i(A)$ *be the coefficient of* x^{n-i} *in* $\phi(A)$*. Here the characteristic map* $ch : \overline{N} \to \mathbb{C}^{n-1}$ *has been defined as* $ch(A) := (\phi_2(A), ..., \phi_n(A))$*. This* $\bar{N}$ *is nothing but our* $SL(n)\mathfrak{r}(P_\sigma)$*. As is proved in Corollary 5.6, this is independent of the choice of* P_σ*. The characteristic map ch above coincides with the composed map*

$$SL(n)\mathfrak{r}(P_\sigma) \subset \mathfrak{sl}(n) \to \mathfrak{h}/W.$$

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